LST - Lecture 6

Controllability and Observability

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VI. Observability and Controllability

1. Pseudo-states ξ and their function space
2. Effect of unimodular matrices on this function space
3. States associated with the pseudo-state
4. State-space trajectories
5. Controllability
6. Necessary and sufficient conditions for controllability.
7. Observability
8. Reachability and i.d. zeros
9. Observability and o.d. zeros
   (nonobservability and g.c.l.d.)

Recap:

PMD (1) (poly. matrix descr.)
State-space (2)
Matrix Function (3)

(1) **PMD:**

$$P = \begin{bmatrix} T & U \\ -V & W \end{bmatrix} \qquad \begin{cases} T\xi + Uu = 0 \\ -V\xi + Wu = y \end{cases}$$

$T \in \mathbb{R}(s)^{r \times r}$

$\dim u = p \quad \dim y = q.$

$$\boxed{G(s)} = VT^{-1}U + W$$

② State-space:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$B \in \mathbb{R}^{n \times p}$
$C \in \mathbb{R}^{q \times n}$
$A \in \mathbb{R}^{n \times n}$

$$\boxed{G(s)} = C(sI - A)^{-1}B + D$$

③ Matrix fractions

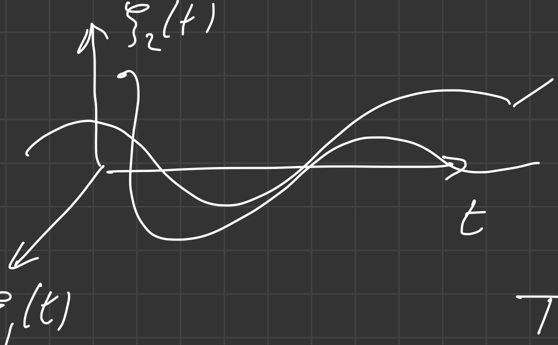
$$\boxed{G(s)} = \bar{N}\bar{M}^{-1} = \bar{N}\bar{D}^{-1}$$
$$\bar{N}(s)\bar{M}(s)^{-1} = \bar{N}(s)\bar{D}(s)^{-1}$$

right fraction

$$= M(s)^{-1}N(s) = D(s)^{-1}N(s)$$

left fraction.

1.) Pseudo-state ξ



$\psi_1(t) \in \mathbb{R}^{r \times 1}$

$\psi_2(t) \in \mathbb{R}^{r \times 1}$

$\vdots$

$$T(s)\xi = 0$$

zero input to the PMD.

$$T\left(\frac{d}{dt}\right)\xi = 0$$

o.d.e.

Let $\{\psi_k(t)\}_{k=1}^{n}$ denote a basis of the vector space $\mathcal{X}$.

Let $x(0) = \begin{bmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{bmatrix}$ be any vector of $\mathbb{R}^n$

$\forall \xi \in \mathcal{X}$ there exists a unique $x(0) \in \mathbb{R}^n$

such that

$$\xi(t) = \psi(t)\, x(0), \quad \forall t > 0^-$$

the map $x(0) \in \mathbb{R}^n \longmapsto \{\xi(\cdot)\} = \{\psi(t)\, x(0)\} \in \mathcal{X}$

is a $\mathbb{R}$-linear bijection (isomorphism) between $\mathbb{R}^n$ onto $\mathcal{X}$.

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2. Effect of unimodular matrices on the solution function space of the pseudo-state ξ.

Let $T(s) \in \mathbb{R}(s)^{r \times r}$ be nonsingular and $L \in \mathbb{R}(s)^{r \times r}$ be unimodular.

$$\bar{T} \triangleq L \cdot T$$

Consider the diff. equations

$$T\left(\frac{d}{dt}\right)\xi = 0 \qquad t \geq 0$$

$$\bar{T}\left(\frac{d}{dt}\right)\bar{\xi} = 0 \qquad t \geq 0$$

with solution spaces X, $\bar{X}$

L unimodular $\Longrightarrow$ $X = \bar{X}$

(unimodular matrices do not alter the solution space)

<u>Proof:</u> Since L is unimodular L^{-1} and L are polynomial matrices

$$T\xi = 0 \Longrightarrow LT\xi = \bar{T}\xi = 0$$

(if ξ satisfies T, it also satisfies $\bar{T}$)

$$\bar{T}\bar{\xi} = 0 \Longrightarrow L^{-1}\underbrace{\bar{T}\bar{\xi}}_{0} = T\bar{\xi} = 0$$

(if $\bar{\xi}$ satisfies $\bar{T}$, it also satisfies T).

3. State associated with a pseudo-state trajectory and constructing ψ_k's., a basis of $\mathcal{X}$.

→ perform a "canonical" realization of $T(\frac{d}{dt})\xi = 0$

Step 1 | Reduce to hermite "canonical" form
$T(\frac{d}{dt})\xi = 0$, $T(s)\xi = 0$
(Similar to the Smith form, but only reducing so as to introduce zeros below the main diagonal of $\overline{T}(s)$).

Step 2 | find solutions to the diagonal elements, scalar o.d.e$_s$.

step 3 | proceed inductively using the previous solutions by solving a forced o.d.e.

an example will clarify this step.

Example: $T(s)\xi = 0$

$$\begin{bmatrix} s+1 & s+1 \\ 0 & \boxed{(s+2)^2} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0$$

Step 2 | $\frac{d}{dt}\xi_1 + \xi_1 = 0 = \dot{\xi}_1 + \xi_1$

$$\xi_1(t) = e^{-t} \qquad \{\psi_1(t)\} = \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}$$

$$\boxed{0 = (s+2)^2 \xi_2}$$

$$\ddot{\xi}_2 + 4\dot{\xi}_2 + 4\xi_2 = 0$$

(1) $\xi_2(t) = e^{-2t}$ (2) $\xi_2(t) = t e^{-2t}$

Step 3 Solve $\dot{\xi}_1 + \xi_1 + \dot{\xi}_2 + \xi_2 = 0$ for ξ_1

using each one of the two ξ_2

(1) for $\xi_2 = e^{-2t}$

$$\dot{\xi}_1 + \xi_1 + \frac{d}{dt}\left(e^{-2t}\right) + e^{-2t} = 0$$

$$\xi_1(t) = e^{-t} - e^{-2t}$$

$$\psi_2(t) = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix}$$

(2) $\dot{\xi}_1 + \xi_1 + \dot{\xi}_2 + \xi_2$

$$\xi_2 = t \cdot e^{-2t}$$

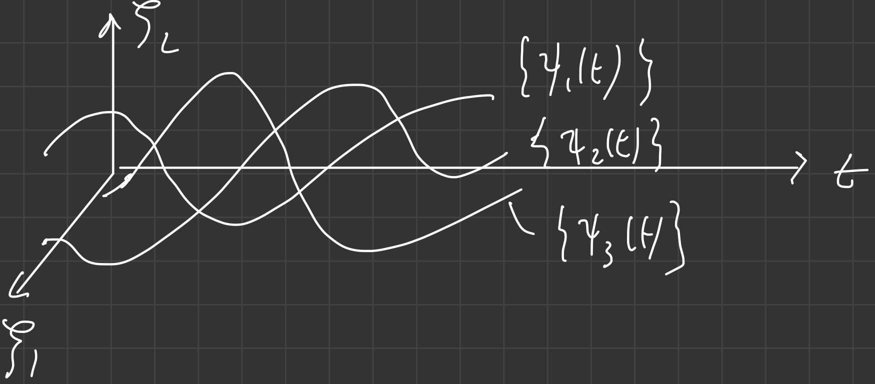
$$\dot{\xi}_1 + \xi_1 + \frac{d}{dt}\left(te^{-2t}\right) + te^{-2t} = 0$$

$$\xi_1 = -t e^{-2t}$$

$$\psi_3(t) = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -te^{-2t} \\ te^{-2t} \end{bmatrix}$$

$\{\{\psi_1(t)\}, \{\psi_2(t)\}, \{\psi_3(t)\}\}$ is a basis of the vector space of trajectories

$$\operatorname{span}\{\{\psi_i(t)\}\} = \mathcal{X}.$$



4. State-space trajectories

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Let $u(\cdot)$ be a given input, let us compute the solution, (trajectory),

$$\{x(x_0, t, u(\cdot))\}.$$

First, consider without input, $u = 0$

$$\dot{x} = Ax$$

$$x(t) = e^{At} \cdot x_0$$

$$e^{At} \triangleq I + At + \frac{A^2t^2}{2} + \cdots + \frac{A^k t^k}{k!} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{A^i t^i}{i!}$$

<u>Thm</u>:

$$e^{At} = \beta_0(t) I + \beta_1(t) A + \cdots \beta_{n-1}(t) A^{n-1}$$

Cayley–Hamilton theorem:

$$\lambda(A) = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_n$$

$$\lambda(A)(A) = A^n + \alpha_1 A^{n-1} + \alpha_2 A^{n-2} + \cdots + \alpha_n I = 0$$

use $$A^n = -\alpha_1 A^{n-1} - \alpha A^{n-2} - \cdots - \alpha_n I$$

in $e^{At} = \sum_{i=1}^{\infty} \frac{A^i t^i}{i!}$ to get $\beta_0, \cdots, \beta_{n-1}$

<u>Example</u>

$$\boxed{e^A = \beta_0 A + \beta_1 I} \qquad \dim A = 2.$$

<u>Thm</u>: such an equation is also valid on the spectrum of A.

i.e. $$e^{\lambda_1} = \beta_0 \lambda_1 + \beta_1$$

$$e^{\lambda_2} = \beta_0 \lambda_2 + \beta_1$$

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \qquad \lambda(A) = (\lambda+1)(\lambda-6)$$

$$\begin{cases} e^6 = \beta_0 6 + \beta_1 \\ e^{-1} = \beta_0(-1) + \beta_1 \end{cases}$$

$$\beta_0 = \frac{e^6}{7} - \frac{e^{-1}}{7}$$

$$\beta_1 = \frac{e^6}{7} + \frac{6}{7} e^{-1}$$

→ <u>Thm:</u> $e^{At}B$ becomes a linear (time varying) linear combination of

$$\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

$$e^{At}B = \alpha_1(t)B + \alpha_2(t)AB + \cdots \alpha_n(t)A^{n-1}B$$

$$\dot{x} = Ax + Bu$$

what is $\{x(t, x_0, u(\cdot))\}$?

the solution trajectory.

$$\frac{d}{dt}\left(e^{At}\right) = Ae^{At} = e^{At}A$$

$$A\left(I+At+\frac{A^2t^2}{2}+\cdots\right) = \left(I+At+\frac{A^2t^2}{2}+\cdots\right)A$$

$$\frac{d}{dt}\left(e^{-At}\cdot x\right) = \frac{d}{dt}\left(e^{-At}\right)x + e^{-At}\cdot\dot{x}$$

$$= -Ae^{-At}x + e^{-At}(Ax+Bu)$$

$$= -A\cancel{e^{-At}}x + A\cancel{e^{-At}}x + e^{-At}Bu$$

$$= e^{-At}Bu$$

$$\int_0^t \frac{d}{d\tau}\left(e^{-A\tau}x\right)d\tau = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$e^{At}\left(e^{-At}x(t) - \underbrace{e^{-A0}}_{I}\cdot x(0)\right) = e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}Bu\,d\tau$$

$$\boxed{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}$$

5. Controllability

Definition: $\dot{x} = Ax + Bu$

The pair (A, B) is said to be controllable if for any initial state and any final state, there exists an input that transfers x_0 to x_1 in finite time.

6. Necessary and sufficient conditions for controllability

Theorem: Equivalent statement:

1. The n dimensional system $\dot{x} = Ax + Bu$ is controllable according to its definition.

2. The $n \times n$ matrix

$$W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau}$$

$$= \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau$$

is nonsingular for any $t > 0$

3. The $n \times p$ controllability matrix

$$\mathcal{C} = \left[B \;\; AB \;\; \cdots \;\; A^{n-1}B \right]$$

has rank n.

4. The $n \times (n+p)$ matrix

$$\left[A - \lambda I \;\vdots\; B \right]$$ has full rank

at every eigenvalue λ of A.
(if λ is complex, then gaussian elimination with complex multiplication should be used).

5. If in addition all eigenvalues of A have negative real parts, then the unique solution of

$$A W_c + W_c A^T = - B B^T$$

is positive definite solution W_c.
The solution is called the controllability grammian

$$W_c = \int_0^\infty e^{A\tau} B B^T e^{A^T\tau} \, d\tau$$

$\boxed{1. \Longleftrightarrow 2.}$

$$\int_{\tau=0}^{t} e^{-A(t-\tau)} BB^T e^{-A^T(t-\tau)} \, d\tau$$

$$\bar{\tau} = t - \tau$$

$$\int_{\bar{\tau}=t}^{0} e^{-A\bar{\tau}} BB^T e^{A^T\bar{\tau}} (-d\bar{\tau})$$

$$= \int_0^t e^{\tau} BB^T e^{A^T\tau} \, d\tau$$

$$\Longrightarrow \quad W_c \geq 0$$

if is $W_c > 0 \iff |W_c| \neq 0$

(W_c is nonsingular)

$|W_c| \neq 0 \implies$ controllability

$$\boxed{x(t_1) = e^{At_1} x(0) + \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau) \, d\tau}$$

We claim that the input $\quad x_1 = x(t_1)$

$$u(t) = -B^T e^{A(t_1-t)} W_c^{-1} \left[e^{At_1} x_0 - x_1 \right]$$

achieves transfer from x_0 to x_1.

$t = t_1$

$$e^{At_1} x(0) + \int_0^{t_1} e^{A(t_1-\tau)} (-BB^T) e^{A^T(t_1-\tau)} \, d\tau \; W_c^{-1} \left(e^{At_1} x_0 - x_1 \right)$$

$$= e^{At_1} x_0 - W_c W_c^{-1} \left(e^{At_1} x_0 - x_1 \right) = x_1$$